Abstract—The phase of complex signals is wrapped since it can only be measured modulo-2π; unwrapping searches for the 2π-combinations that minimize the discontinuity of the unwrapped phase, as only the unwrapped phase can be analyzed and interpreted by further processing. Given an estimate of the phase gradient (i.e., of the instantaneous frequency), the 2-D unwrapped phase can be obtained as a solution of a variational problem. The analysis of unwrapping is done quite separately from instantaneous frequency estimation so that the reliability problem. The analysis of unwrapping is done quite separately from instantaneous frequency estimation so that the reliability of both steps can be assessed independently. Various methods for evaluating 2-D instantaneous frequency are presented and compared in the presence of noise and amplitude variations. A study has also been made on aliasing arising in areas where, with respect to instantaneous frequency, spatial sampling is insufficient. The presence of noise in the data further complicates phase aliasing analysis since there is no way to distinguish between the aliasing due to noise or that due to steep phase slopes.

I. INTRODUCTION

PHASE is measured modulo-2π and its values are then called principal values (PV). When dealing with the linear processing of a sampled phase the PV's have to be unwrapped: this results in a "continuous" function, the 2π discontinuities being eliminated or, at least, reduced. This paper first presents an approach for phase unwrapping in a discrete twodimensional (2-D) domain, and then gives an analysis of the main limitations that are encountered in practice.

The importance of phase information is fundamental to signal processing in that both homomorphic signal processing and homomorphic deconvolution require unwrapped phase estimates. When dealing with nonstationary processes the unwrapped phase and its first derivative or instantaneous frequency, spatial sampling is insufficient. The presence of noise in the data further complicates phase aliasing analysis since there is no way to distinguish between the aliasing due to noise or that due to steep phase slopes.

The compensation of lens aberration calls for the estimation of wavefront distortion from phase difference measurements. This led to investigations into the problems of 2-D unwrapping from phase differences. References [10], [15] introduced the fundamentals of least mean square (LMS) unwrapping. In traveltime tomography unwrapped phase information is needed for diffraction tomography using Rytov approximation [8]. Furthermore 2-D phase unwrapping techniques have also been proposed for the correction of magnetic field nonuniformity in magnetic resonance imaging [7]. In synthetic aperture radar [9], the interferometrical phase fringes obtained from the combination of two coherent narrowband images have to be unwrapped to obtain the terrain elevation [14], [21]. In texture discrimination and image segmentation, local phase information comes from the unwrapping of the phase image obtained by the application of 2-D Gabor filters [6].

In general, any ambiguity that arises in phase unwrapping is the result of poor sampling of rapid phase variations. Different approaches are needed for the unwrapping of 1-D and 2-D phases [24]. Conventional 1-D phase unwrapping algorithms integrate the wrapped phase difference between two contiguous points or adapt such integration steps [26]. Extending 1-D algorithms to contiguous or crossing 1-D slices of 2-D phase measurements does not generally yield satisfactory results since propagation and accumulation of unwrapping errors generate streaks along the slice directions. The quality of 2-D unwrapping using this approach depends on the strategy adopted for 1-D slicing, as well as on 2-D phase sampling and its signal to noise ratio (SNR).

In general, the complex signal z(x) in 2-D space x = (x,y) is

\[ z(x) = |z(x)| \exp \{j \angle z(x)\} = |z(x)| \exp \{j \angle \psi(x)\} = a(x) \exp \{j \angle \phi(x)\} + w(x) \]  

(1)

the noise w(x) is a zero-mean complex random signal, amplitude a(x) and phase \( \phi(x) \) variations are unknown and \( \angle \) denotes the argument. The term \( \angle \psi(x) \) indicates the principal value (PV) of the unwrapped and unknown phase function \( \psi(x) \) that, in absence of additive noise, should be \( \angle \psi(x) = \angle \phi(x) \). The PV is a nonlinearity that transforms the unwrapped phase \( \psi(x) \) into the wrapped phase given as \( \angle z(x) \). Even if, for congruency, \( \angle z(x) = \angle \psi(x) \), should hold, a formal distinction will be made for the sake of clarity: \( \angle z(x) \) represents the wrapped phase while \( \angle \psi(x) \) is obtained by applying the nonlinearity PV to the continuous unwrapped phase \( \psi(x) \). Since \( \angle z(x) \) is given or measured, algorithms for phase unwrapping are used in an effort to retrieve the phase
surface $\psi(x)$ and to approximate $\phi(x)$ from the knowledge of $z(x)$, regardless of any amplitude variations $a(x)$. This is the same as searching for the $2\pi$ shift combination that produces a regular phase curve $\psi(x) = \phi(x) + 2\pi n(x)$, where $n(x)$ is an integer. The estimation of phase gradient or instantaneous frequency $f^w(x) = \nabla \phi(x) = \partial z(x)/\partial x$ directly from a sampled version of $z(x)$ seems an alternative approach to avoid phase unwrapping. However, once IF’s has been estimated, the unwrapped phase can be obtained directly from the integration of those estimates. The approach herein presented is similar to the reconstruction of surfaces from indirect measurements of local slopes in vision systems (see e.g. [5] for additional references) thus having a discontinuous measurement such as PV phase. This paper shows also that IF estimation is only an intermediate step in unwrapping and that there exists a strict correlation between the reliability of phase unwrapping and IF estimation technique. Three aspects of 2-D phase processing are discussed separately: 1) unwrapping from IF estimates; 2) techniques for estimating the IF’s; 3) the limitations due to noise and phase sampling.

After a brief description of the extension of 1-D unwrapping to the 2-D domain (1-D slicing) a 2-D least mean square (LMS) unwrapping algorithm from integration of local IF is proposed (Section II). Other LMS approaches [10], [15] minimize the mean square error between unwrapped and wrapped phase differences; the approach proposed here separates the estimation of IF from the LMS unwrapping thus leading to a full generalization of LMS unwrapping. Since unwrapping corresponds to 2-D linear filtering of IF, estimates the reliability of phase unwrapping depends, in principle, on the technique used for the IF estimation from the sampled complex signal $z(x)$ provided that SNR is high enough to avoid threshold effects (e.g., local cycle skips). Section III discusses techniques for 2-D IF estimation and Cramer-Rao (CR) lower bounds, threshold effects and limitations in phase unwrapping. In addition, we propose a coherent filtering useful to limit the influence of amplitude variations in IF estimation (e.g., in synthetic aperture radar interferometry) and we compare its reliability with that of constant amplitude examples. Phase aliasing, a further limitation to IF estimation as well as to phase unwrapping, is discussed in Section IV. With respect to 1-D phase, 2-D phase measurements have the advantage that aliasing areas in some cases could, in principle, be identified. Steep phase slopes and noise, the main causes of phase aliasing, are discussed separately and their joint effects are compared. Section V shows simulations of 2-D phase unwrapping by comparing the theory and methods for IF estimation with the results of LMS unwrapping. An application of phase unwrapping to synthetic aperture radar (SAR) interferometry is also discussed.

II. PHASE UNWRAPPING

A discussion is given first of 2-D phase unwrapping as an iterative 1-D slicing and then of 2-D unwrapping from the IF estimation. All the signals referred to herein are sampled.

A. 2-D Unwrapping by 1-D Slicing

If the unwrapped phase $\psi(x)$ were known, the IF for any two close points $x_i$ and $x_{i+1}$ could be evaluated as the finite difference (FD)

$$f_i(x_i) = \frac{\psi(x_{i+1}) - \psi(x_i)}{|x_{i+1} - x_i|} 1_i,$$  \hspace{1cm} (2)

where $1_i = (x_{i+1} - x_i)/|x_{i+1} - x_i|$ is a unit vector along the 1-D direction of IF. The IF evaluated as PV finite difference (PVFD) of wrapped phase is

$$f_i^{w}(x_i) = \frac{\zeta(x_{i+1})\zeta^*(x_i)}{|x_{i+1} - x_i|} 1_i,$$  \hspace{1cm} (3)

where * indicates complex conjugate. 1D and 2D phase unwrapping algorithms implicitly or explicitly assume that $f_i = f_i^{w}$; hence $\psi(x_{i+1}) - \psi(x_i) = \zeta(x_{i+1})\zeta^*(x_i)$ is a condition that should be verified to avoid phase aliasing.

Since PVFD’s should be independent of the unwrapping and the congruency condition on finite difference $\psi(x_{i+1}) - \psi(x_i) = \zeta(x_{i+1})\zeta^*(x_i)$ should hold, the unwrapped phase can be obtained from the summation of the PVFD’s evaluated for adjacent points along an assigned path (1-D slicing)

$$\psi(x_N) = \zeta(x_1) + \sum_{i=2}^{N} \zeta(x_i)\zeta^*(x_{i-1}).$$  \hspace{1cm} (4)

In the unwrapped phase $\psi(x_N)$ the ambiguity due to an overall constant phase shift of multiples of $2\pi$ is irrelevant. Unwrapping algorithms for 1-D phase [26] generally retrieve the continuous phase information by summing the PVFD’s as (4) and any incorrect estimation of IF’s from PVFD’s due to noise and phase aliasing leads to the propagation of phase unwrapping errors. Unfortunately the identification of phase aliasing for 1-D phase data is not feasible without some a priori information. The 2-D unwrapping by 1-D slicing is dependent on the summation path and this dependency makes its reliability questionable [7]. In 2-D, unwrapping errors due to aliasing appear as easily identifiable streaks along the summation paths due to conflicting solutions if the path strategy is not properly controlled [12].

B. 2-D Unwrapping from IF

There is a definite advantage in dealing with 2-D unwrapping as a whole problem instead of 1-D slicing algorithm. The 2-D unwrapping algorithm derived here is obtained by the minimization of the difference between the IF $f^w(x,y)$ measured or estimated from complex signal $z(x,y)$ and the IF corresponding to the unknown unwrapped phase: $f(x,y) = \nabla \psi(x,y)$. The derivation of this proposed approach takes advantage of other algorithms of least mean square (LMS) unwrapping that are based on the minimization of FD of unwrapped phase and PVFD for wrapped phase [10], [15], [16]. The advantage of the LMS unwrapping discussed here is that it is no way depends on how the IF’s were evaluated from $z(x,y)$.

Let the IF estimates be

$$f^w(x,y) = f^w_x(x,y)1_x + f^w_y(x,y)1_y$$  \hspace{1cm} (5)
where \( \mathbf{1}_x \) and \( \mathbf{1}_y \) are unit vectors along the \( x \) and \( y \) axes; and let the unwrapped phase \( \psi(x, y) \) be obtained from the minimization of the mean square error between the gradient of unwrapped phase and \( f^{(w)}(x, y) \)

\[
\min_{\psi(x, y)} \left\{ \sum_{(x, y)} \left| f^{(w)}(x, y) - f(x, y) \right|^2 \right\}.
\]  

(6)

Since no \( a \) priori assumptions are made on the unwrapped phase \( \psi(x, y) \), or on \( f^{(w)}(x, y) \), the phase unwrapping solution according to (6) is general. Its minimization corresponds to a variational surface reconstruction problem [5] and leads to the partial differential equation (PDE) for the unwrapping

\[
\nabla^2 \psi(x, y) = \frac{\partial f^{(w)}_x(x, y)}{\partial x} + \frac{\partial f^{(w)}_y(x, y)}{\partial y}.
\]  

(7)

When the phase has been properly sampled with respect to steep phase slopes the LMS estimation leads to a solution congruent with the wrapped phase \( ([\psi]_p \equiv \{z\}) \). From (7) it follows that only the irrotational components of \( f^{(w)}(x, y) \) can be uniquely fitted by a continuous phase surface \( \psi(x, y) \). Also 2-D unwrapping by 1-D slicing algorithm applied along any path becomes reliable as only an irrotational \( f^{(w)}(x, y) \) guarantees the independence of the unwrapped phase from the integration path.

The integration of (7) requires boundary conditions. As the IF's are known even at the boundary the Neumann boundary condition is used in the integration of PDE, the unwrapped phase with the Neumann condition having a constant phase slope. However any error in IF estimation at the boundary (e.g., phase aliasing) influences the results of the unwrapped phase. Sometimes unwrapped phase is known at the boundary or the whole phase image can be partitioned into regions, in both cases Dirichlet or mixed boundary conditions can be used.

A general method to solve elliptic equations is the successive over-relaxation technique that guarantees convergence in approximately \( 2N \) iterations for a square domain of \( N^2 \) nodes. Equation (7) with Neumann condition can also be solved in the transformed domain as 2-D filtering of the IF (uppercase indicates Fourier transformed variables and \( (k_x, k_y) \) are the wavenumbers of the space variables \( (x, y) \))

\[
\Psi(k_x, k_y) = \frac{-j k_x}{k_x^2 + k_y^2} F^{(w)}_x(k_x, k_y) + \frac{-j k_y}{k_x^2 + k_y^2} F^{(w)}_y(k_x, k_y).
\]  

(8)

In the Fourier domain 2-D phase unwrapping is considerably more efficient than iterative techniques, though attention must be paid to 2-D data periodicity to guarantee that boundary conditions are satisfied (see, e.g. [25] for a discussion on unwrapping in Fourier domain using minimization of phase differences); a further advantage arises from the computation efficiency of Fast Fourier Transform (FFT). There are applications, such as interferometrical imaging (see Section V) where the unwrapped phase is known, or can be reliably estimated in selected areas. When we have mixed Neumann and Dirichlet boundary conditions, or we want to use any weighting factors in unwrapping [13], [26] to achieve the solution for the overall image, PDE should be solved using the more costly iterative techniques.

## III. IF ESTIMATION

Let us turn to the estimation of IF samples. We consider a sequence of noisy observations

\[
z_{lm} = z(x_l, y_m) = a_{lm} \exp \{j \psi_{lm}\} + w_{lm}
\]  

(9)

the sampling being indexed as \( x_l = 1 \Delta x \) and \( y_m = m \Delta y \). Without loss of generality, unitary and uniform space sampling is assumed: \( \Delta x = \Delta y = 1 \). The algorithms for 2-D IF estimation can be adapted from 1-D frequency estimation techniques [3], [4]. In this section, only some of the most advantageous algorithms for 2-D phase unwrapping are derived, analyzed and compared.

### A. Principal Value Finite Difference (PVFD)

The IF estimation using two points only corresponds to the PVFD estimator given by (3)

\[
f^{(w)}(x_1, y_m) = z_{l+1,m} z_{l,m}^* \mathbf{1}_x + z_{l,m+1} z_{l,m}^* \mathbf{1}_y.
\]  

(10)

The analysis of (7) shows that PVFD (10) leads to the unwrapping by phase differences minimization [10], [15]. Given its simplicity, the PVFD has been a useful IF estimation method in LMS unwrapping with good SNR.

In the presence of noise, any filtering of \( z_{lm} \) could be performed within its effective 2-D bandwidth. However, since \( z_{lm} \) represents a 2-D phase-modulated signal, any filter designed to reduce noise should necessarily have a large bandwidth to preserve the phase information. The weighted averaging of PVFD's (10) represents an alternative way to signal filtering. Reference [17] presented an efficient weighting for PVFD's that attains CR bounds for high SNR, the computational advantages of this weighted averaging of PVFD's is attractive in 2-D phase unwrapping. A summary of the statistical properties of generalized PVFD's can be found in [18].

### B. Complex Signal Phase Derivative (CSPD)

Given the signal \( z_{lm} \), the IF is estimated by the 2-D generalization of the uncoherent technique for frequency demodulation in communication systems [28]. Taking separately the real \( (u_{lm}) \) and imaginary \( (v_{lm}) \) part of \( z_{lm} \) it follows the IF estimation evaluated from complex signal

\[
f^{(w)}(x_l, y_m) = \frac{\partial z_{lm}}{\partial x_l} \mathbf{1}_x + \frac{\partial z_{lm}}{\partial y_m} \mathbf{1}_y
\]

\[
= \frac{1}{|z_{lm}|^2} \left\{ \left( u_{lm} \frac{\partial u_{lm}}{\partial x_l} - v_{lm} \frac{\partial u_{lm}}{\partial y_m} \right) \mathbf{1}_x \right. 
\]

\[
+\left. \left( u_{lm} \frac{\partial v_{lm}}{\partial y_m} - v_{lm} \frac{\partial u_{lm}}{\partial x_l} \right) \mathbf{1}_y \right\}.
\]  

(11)

FIR filters or FFT's are used for evaluating the derivatives within the bandwidth of \( z_{lm} \). Low values of \( |z_{lm}| \) give rise to values of IF higher than \( \pm \pi \) and this unavoidably leads to an
IF dominated by noise. A reasonable nonlinear estimator for each component of IF that limits the influence of outliers in phase unwrapping is

$$G[f^{(w)}] = \begin{cases} f^{(w)} & |f^{(w)}| < \eta \\ \eta \text{sign}[f^{(w)}] & |f^{(w)}| > \eta \end{cases}$$

where $\eta$ indicates a threshold that depends on data SNR. $\eta$ is the value of the IF above threshold. Isolated values of IF larger than $\pm \pi$ could be caused by noise resulting in a noisy unwrapped phase. Applications has shown that zero tapering ($\eta = 0$) of high IF values has the advantage of reducing the noise at the expense of a loss of dynamics in unwrapped phase for high IF values while for 2-D unwrapping seems more advantageous to use a limiter.

C. Maximum-Likelihood (ML)

The Maximum Likelihood (ML) frequency estimator is well known to be given by the location of the peak of the signal periodogram [22]. The 2-D signal is assumed to have a constant but unknown amplitude $a_{lm}$ and IFs $\phi_{nm} = f_x l + f_y m + \phi_0$ with a superimposed Gaussian noise $w_{lm}$. The ML estimator of IF at the sample $(x_l, y_m)$ can be obtained by maximizing, with respect to the IF's $f_x$ and $f_y$, the 2-D discrete Fourier transform of $N_x \times N_y$ observations centered at the sample $(x_l, y_m)$

$$\max_{f_x, f_y} \left\{ \sum_{i_x=-(N_x-1)/2}^{m+(N_x-1)/2} \sum_{i_y=-r(N_y-1)/2}^{m+(N_y-1)/2} z_{i_x,i_y} \exp \{-j(f_x i_x + f_y i_y)\} \right\}. \quad (13)$$

The 2-D FFT is a simple way to implement a “coarse” search; for the “fine” search any optimization method can be used provided that the “coarse” IF estimates fall within the main lobe of the 2-D periodogram. Usually the granularity of the FFT bins is much larger than the square root of the main lobe of the 2-D periodogram. The computational load required for attaining the lower bound is prohibitive, even using 2-D FFT. The search strategies for the maximum are not relevant to this paper and the reader is referred to [1] for discussions on the use of Newton search in 1-D IF estimation as well as the choice of “coarse” FFT bins.

A reduction in IF variance is expected when the data matrix $N = N_x \times N_y$ is increased, provided that the frequency remains constant. In the case of real 2-D phase data such a reduction depends on the stationarity of the local frequency. Frequency estimation based on three samples (i.e., PVFD) represents an extreme solution for applications with nonstationary phase gradient. In unwrapping, ML IF estimates with large $N$ values allow the separation of the average phase information from the details, while the PVFD of the residuals filtered within the signal bandwidth allows the estimation of local changes.

D. Cramer-Rao Bounds

The Cramer-Rao (CR) bound represents a lower limit for the variance of IF estimators that is met by ML when the SNR is high enough. Assuming the model (9) with constant IFs and amplitude $a_{lm}$, let us consider, for instance, the estimates of $f_x$. 2-D ML (13) corresponds to a sequence of $N_x$ IF estimates of $N_y$ points each. For each of these estimates the CR bound depends on SNR ($\gamma = E[a_{lm}^2]/2E[w_{lm}^2]$) [22]: $6/7 N_x(N_x^2 - 1)$. Even if the phase of each of these $N_x$ points is linearly related to the neighboring ones the IF estimate is independent of the phase value. Therefore each estimate of $N_x$ points is independent of the other estimates; the lower variance for $N_x$ estimates of $f_x^{(w)}$ becomes

$$\text{var}[f_x^{(w)}] = \frac{6}{\gamma N(N_x^2 - 1)} \quad \text{(14)}$$

in agreement with Kay’s relationship [17]; similarly for $f_y^{(w)}$

$$\text{var}[f_y^{(w)}] = \frac{6}{\gamma N(N_y^2 - 1)}, \quad \text{(15)}$$

In the case of low SNR the noise dominates the IF estimates and a threshold effect occurs in IF bounds (Appendix A).

E. Maximum-Likelihood IF Estimation in the Presence of Amplitude Variations

In the presence of amplitude modulation and noise, low amplitudes of the signal $z_{lm}$, are quite likely. Threshold occurs at high SNR. Threshold can be remarkably reduced if the IF estimates are performed in two separate steps (each step is performed on the overall data).

- **Narrowband or coherent filtering**: the signal $z_{lm}$ is down-converted using ML estimated IF's, then it is averaged over $N_x \times N_y$ samples the only purpose being to limit bandwidth spreading due to amplitude modulation but to still preserve the information content of the signal (i.e., 2-D phase modulation); the signal is then re-modulated using the same IF's previously estimated;

- **ML IF estimation**: the IF's are estimated using the ML approach but after coherent filtering.

Basically, threshold depends on the bandwidth of narrowband filtering (i.e., dimensions of $N_x \times N_y$). Data decimation sometimes becomes a necessary intermediate step for a reduction of computational load at the expense of a moderate loss of performance.

F. Simulations

Simulations here are performed in order to compare the performance of the PVFD (10), CSPD (11) and ML (13) IF estimators with respect to CR bounds and to analyze the threshold effects in 2-D phase unwrapping. In the signal model $a_{lm}$ is either constant or a random variable independent with noise and phase (Figs. 1 and 2, respectively). For each value of SNR 200 independent experiments were carried out with $\phi_{lm} = 2\pi 0.05(1 + m)$. The IF root mean square error (RMSE) was then compared with CR bounds and with the evaluation of ML threshold given in Appendix A.
Fig. 1. Performance of 2-D IF estimators versus SNR, PVFD given by (10) with \( N = 32 \); CSPD given by (11) (+); ML with \( N = 9, 25, \) and 49 samples (\( \times \)) and \( N_x = N_y = \sqrt{N} \). CR lower bounds (dashed line) with threshold effects described in Appendix A are also indicated.

Fig. 2. Performance of ML IF estimator of signal with random amplitude variation (exponential pdf) with \( \beta \) and \( \alpha \) and \( N_x = N_y = \sqrt{N} \). ML estimate (solid line) and coherent amplitude filtering before ML estimate (dashed line).

Fig. 1 shows that PVFD’s (10) and CSPD (11) behave approximately in the same way for \( SNR > 3 \) dB. A limiter (12) is necessary in CSPD for \( SNR < 3 \) dB since outliers in IF’s can lead to a wrong unwrapped phase. ML estimators using small data matrices \( (N_x = N_y = \sqrt{N} = 3, 5, \) and 7) achieve the CR bounds (14) or (15) for high SNR values.

The dashed line here represents also the quantification of 2-D threshold effects in the ML IF estimator obtained by adapting the evaluation of threshold for 1-D IF estimation as described in Appendix A (threshold for PVFD is negligible). Threshold occurs at \( SNR = 0 \) dB for \( N = 9 \), \( SNR = -3 \) dB for \( N = 25 \) and \( SNR = -5 \) dB for \( N = 49 \). Being a phase–modulated signal, the bandwidth of \( z_{lm} \) is larger than (or at least equal to) the bandwidth of phase \( \phi_{lm} \), therefore in the presence of uncorrelated noise the ML performs better than IF estimation by any other method, even if \( z_{lm} \) has been low pass filtered within its bandwidth. This is a strong argument in favor of ML IF estimation for 2-D phase unwrapping, provided that the patch size \( N_x \times N_y \) is comparable with the stationarity of the phase gradient.

Fig. 2 shows the RMSE obtained assuming a random amplitude with exponential pdf \( p_x(a) = \exp \left( -\alpha a / \sigma_a \right) / \sigma_a \) with \( \alpha > 0 \). This case is relevant for synthetic aperture radar interferometry (see Section V). SNR is now defined as \( \gamma = \sigma_e^2 / \sigma_w \). Because of random amplitude modulation, the RMSE of the ML IF estimates (solid line) is higher than the corresponding simulations with constant amplitude (Fig. 1). An amplitude lower than the mean value is very likely for an exponential pdf so that threshold occurs at SNR higher than the corresponding constant amplitude. The RMSE after coherent amplitude filtering (dashed line) becomes more effective by increasing the window size \( N \) as this considerably reduces the residual amplitude modulation, thus leading the threshold close to the constant amplitude example (Fig. 1) and making the coherent filtering particularly attractive.

### IV. LIMITATIONS OF LMS UNWRAPPING

Basically, LMS phase unwrapping is a 2-D filtering. The better the IF estimation step is carried out, the better the 2-D phase is unwrapped. However, insufficient phase sampling (phase aliasing) of closely spaced phase fringes (e.g., high IF’s) and the interaction of high IF’s with noise are crucial problems in phase unwrapping. The reliability of the unwrapping algorithms should be compared on the basis of robustness with respect to errors and error propagation within the domain. This section describes the detection of singular points due to aliasing and discusses two limitations of 2-D phase unwrapping: aliasing caused by rapid phase variations and aliasing caused by noise.

#### A. Aliasing: Singular Points

Estimated IF \( f^{(\theta)}(x, y) \) can be separated into rotational and irrotational components: the 2-D phase aliasing is the only cause of the rotational component; the presence of rotational sources, called singular points, identifies the sites of phase aliasing. The summation of the IF along an arbitrary closed path depends on the rotational component \( \int f^{(\theta)}(x, y) \cdot dl = \sum_i \Sigma_j \Sigma_k \cdot 2\pi m_i, \) where \( m_i \) is the multiplicity of each singular point enclosed in the path [23]. This relationship is useful for detecting, from the wrapped phase, isolated singular points within the domain (the 4 point mesh allows the detection of multiplicity \( m_i = \pm 1 \)). Unlike 1-D unwrapping where aliasing leads to uncontrolled error propagation, singular points can be easily detected from the 2-D wrapped phase using, for instance, PVFD. Thus phase aliasing is identified from IF using indirect methods.

#### B. Aliasing Caused by High IF

Between two isolated singular points of opposite multiplicity there could be a whole area delimited by a line of high IF’s connecting the singular points (aliasing line) where the IF evaluated from the wrapped phase has been aliased [24]. Along the aliasing line the “true” IF is \( |f| > \pi \) while the IF...
C. Aliasing Caused by Noise

The phase aliasing problem is complicated by a low SNR as this increases the number of singular points. Singular points due to rapid phase variations or to low SNR are difficult to distinguish and classify. In addition, the local phase slope influences the singular point distribution caused by noise only.

A general analysis of the interaction between noise and IF values cannot be performed easily because of the nonlinearity of the PV operator. However, in the case of low and high SNRs using PVFD as IF estimation analytical approximations are possible.

Fig. 3 shows the probability of singular points or the probability of 2-D phase aliasing obtained considering four point PVFD summations, assuming a complex signal $z_{lm}$ with linear phase $\phi_{lm} = \Delta \phi \cdot (l + m)$ and uncorrelated Gaussian noise $w_{lm}$. The analytical solution derived in Appendix B for high SNR (dashed line) agrees with simulations (circle) obtained by considering $10^5$ independent experiments. Fig. 3 indicates that by increasing IF ($\Delta \phi$) the number of singular points in an area of constant IF also increases, due to the joint effect of rapid phase variations $\Delta \phi$ and noise.

Uniform phase pdf given by low SNR and high IF's gives a value of probability of phase aliasing equal to $1/3$. In applications, the estimation of singular points density and data SNR is useful both for boundarying areas where the IF estimation can be unreliable and for weighting the LMS unwrapping [13], [24].

Since the PV operator of finite differences in phase is nonlinear (in general $[a + b]_p \neq [a]_p + [b]_p$), the use of a local demodulation scheme to separate the phase trend from the local changes represents a promising solution in some applications where the aliasing occurs for high IF's. In the presence of noise, the IF of the residual phase after the unwrapping should be considered so that, after demodulation and for a given SNR, the IF is zero and the number of singular points is reduced (Fig. 3). Unwrapping after demodulation is the basic idea of iterative unwrapping, the advantage of the demodulation scheme having already been demonstrated by the ML IF estimator.

V. APPLICATIONS

The only way to evaluate the accuracy of the unwrapped phase $\psi(x, y)$ when $\phi(x, y)$ is not known, as generally occur,
is by comparing the PV unwrapped phase \( \psi(x, y) \) and the wrapped phase \( \angle z(x, y) \) (unwrapping congruency). The argument of signal \( z(x, y) \) demodulated by the unwrapped phase is the residual

\[
\angle z(x, y) \exp[-j\psi(x, y)] = \angle z(x, y) - [\psi(x, y)]_{\text{PV}}, \quad (16)
\]

that gives a measurement of congruency. Whenever no singular points are detected in the wrapped phase the LMS unwrapping achieves no residual. Singular points caused by noise give a spatially uncorrelated residual (whitening) while unwrapping achieves no residual. Singular points caused by phase become a useful tool. Similarly to ML IF estimation, where a reduction of the bandwidth of \( z(x, y) \) the wrapped phase \( \angle z(x, y) \) exp \([-j\psi(x, y)] \), thus allowing a noise reduction by low-pass filtering at every step. In fact, despite the simplicity of iterative unwrapping it is effective in severe noisy situations. Only reference is made to it here.

A. Simulations of LMS Unwrapping

Two simulations are presented: these compare the IF estimation techniques in phase unwrapping and show their limitations due to phase aliasing.

Fig. 4 shows 2-D LMS phase unwrapping at SNR = 5 dB. Fig. 4(a) shows the phase model \( \phi(x, y) \); the wrapped phase \( \angle z(x, y) \) is in Fig. 4(b). The maximum value of IF, in the area with closely spaced fringes, is approximately 0.8\( \pi \) and the probability of phase aliasing is close to 1/3 (see Fig. 3). Any phase unwrapping method works properly for most practical applications when the SNR is higher than 10 dB. Fig. 4(c)-(f) compare the unwrapping residuals \( \phi(x, y) - \psi(x, y) \) (scaled to the overall dynamic range) obtained using different IF methods. Fig. 4(e) shows the unwrapping residuals using ML IF estimation with \( N_x = N_y = 3 \) while the residuals of unwrapping using CSPD are in (c). Compare this with Fig. 4(d) where nonlinearity with thresholds \( s = \eta = 2\pi \) has been used [Fig. 4(c)] and (d) have the same scaling]. Large values in \( \phi(x, y) \) due to IF's \( >2\pi \) are strongly limited by the nonlinearity, as shown in Fig. 4(d). Phase unwrapping with PVFD IF is shown in Fig. 4(f) and corresponds to the LMS unwrapping described in the literature [10], [13], [15], [16], [23]–[25]. The residuals using PVFD are mainly concentrated in the closely spaced fringe area indicated by an arrow in Fig. 4(f). In all methods high IF and noise combine to give slowly varying unwrapping residuals.

Note that there is no unique solution for the unwrapping of the aliased phase. However LMS unwrapping has the capability of limiting the propagation of unwrapping errors. For example, Fig. 5(a) shows a phase \( \phi(x, y) \) with an abrupt \( 2\pi \) phase step, an extreme situation representing aliasing due to high IF. Since the discontinuity is exactly \( 2\pi \), the phase step disappears from the analysis of the wrapped phase [Fig. 5(b)]. However, the analysis of the IF's shows an incongruency due to singular points. The residual (16) shown in Fig. 5(d) contains the wrapped \( 2\pi \) step in the wrong location, thus giving an unwrapped phase [Fig. 5(c)] that is smoother around the ambiguous area (given the abrupt phase variations of this example PVFD was used). Compared with the noise-free case [Fig. 5(c)] the overall shape of the unwrapped phase [Fig. 5(e)] is preserved, even when SNR = 3 dB because of negligible contribution of singular dipoles (Appendix B), the corresponding residual is in Fig. 5(f). This example shows that phase with aliasing caused by high IF gives rise to some incongruencies in IF's, that are detected by the presence of singular points. This example shows that, in the presence of phase aliasing, the LMS unwrapping

1) achieves a phase \( \psi(x, y) \) that is smoother than the true solution \( \phi(x, y) \),
2) leaves \( 2\pi \) discontinuities in the residual along paths that connect singular points,
3) preserves the shape of \( \phi(x, y) \) even in the presence of noise.

B. SAR Interferometry

2-D phase unwrapping has recently become of increasing interest due to the use of SAR interferometry for terrain mapping [14]. The LMS unwrapping algorithm is now a standard processing tool, preferred to iterative 1-D slicing [21]. The basic concepts of SAR are briefly illustrated here; the reader is referred to [9] for a complete description of SAR systems. In SAR imaging the amplitude of the complex signal \( I(x, y) \) is proportional to the backscattering of the target (e.g., surface reflection coefficient and relative orientation between radar pointing and target). Since SAR is coherent, the reconstruction of surface phase (i.e., 2-D phase unwrapping) from phase fringes \( \angle I_1(x, y)I_2^*(x, y) \) of two images \( I_1(x, y) \) and \( I_2(x, y) \) obtained illuminating the same object \( x, y \) from two radar locations allows the measurement of surface elevation.

In SAR interferometry 2-D IF estimation and unwrapping is complicated mainly by: 1) amplitude variations, that give a spatially variable SNR; 2) areas not illuminated where missing fringes cannot be recovered in unwrapping; 3) areas of uncorrelation between images \( I_1(x, y) \) and \( I_2(x, y) \) (since images are not collected simultaneously only stationary objects have consistent fringes); 4) speckle noise due to coherent imaging. All these effects can only be classified as phase aliasing areas. Assuming that the amplitude of each signal \( I_1(x, y) \) and \( I_2(x, y) \) has Rayleigh pdf, the pdf of the amplitude of interferometrical image \( I_1(x, y)I_2^*(x, y) \) has an exponential pdf. In the case of random amplitude modulation (Section III) we have shown that the ML IF estimation should be performed after coherent filtering to limit threshold effects.

An example of terrain elevation in the Vesuvius area (Naples, Italy) recovered from 2-D LMS unwrapping of interferometric fringes is shown in Fig. 6. The wavelength is 2.8 cm, the two viewpoints have a distance of 54 m and are approximately located 800 Km from the ground (the eastside looking is shown here). From the amplitude image \( |I_2(x, y)I_2^*(x, y)| \) shown in Fig. 6(a) the shape of the volcano is clearly visible, the black side in the upper left corner indicates an uncorrelated area.
Fig. 5. Unwrapping of aliased phase with an abrupt \(2\pi\) phase step using PVFD's IF estimation: (a) the phase \(\phi(x, y)\); (b) the wrapped phase \(\phi(x, y)\); (c) LMS unwrapping for noise free data; (d) residual (16) of LMS unwrapping; (e) LMS unwrapping for \(\text{SNR} = 3\) dB; (f) residual (16) of LMS unwrapping for \(\text{SNR} = 3\) dB.

VI. CONCLUSIONS

An algorithm for 2-D phase unwrapping that minimizes the mean square error between the estimated instantaneous frequency (IF) and the IF evaluated from the unwrapped phase was arrived at through a generalization of the technique described in the literature. The splitting of IF estimation and LMS unwrapping allowed a separate analysis of the two steps. The LMS unwrapping algorithm discussed here 1) is independent of how the IF is estimated from the complex signal; 2) can be efficiently implemented in the Fourier domain; 3) is general and flexible since it can be adapted to each specific application; 4) is less affected by error propagation and \textit{a priori} assumptions than 1-D slicing algorithms and should be preferred to 1-D slicing for rapid phase variations and SNR below 5 dB.

Phase aliasing is the main limitation in phase unwrapping and arises from insufficient sampling of rapid phase slopes or from low SNR. The ambiguity in 2-D phase unwrapping is clearly identified by the presence of singular points or dipoles in IF's. An additional advantage of LMS unwrapping for low SNR lies in the fact that the influence of singular dipoles is lower than that of singular points.

Most of the 2-D IF estimation techniques discussed here were adapted from the 1-D frequency estimation. The phase-only method for the estimation of IF (PVFD) is reliable for SNR above 5 dB. The IF's and unwrapped phase obtained using CSPD is dominated by outliers for SNR below 5 dB so that a limiter in IF's is needed. When SNR is below 5 dB, ML estimation with a large data window (\(N_x \times N_y = 9\) or more) should be used. The computational disadvantage of ML and the moderate loss of detail with respect to PVFD method makes ML very useful for applications with low SNR and well-behaved IF. This paper proposes an algorithm for 2-D amplitude modulation filtering in ML IF estimation of signals with random amplitude variations, thus making the unwrapping from ML IF estimation feasible for Synthetic Aperture Radar interferometry.

Whenever a smooth IF is a reasonable assumption for a first iteration of unwrapping, 2-D unwrapping of the wrapped residual (i.e., unwrapping after local phase demodulation) becomes less sensitive to noise and retrieves further detail in the phase image since the bandwidth of the phase modulated signal depends on its phase variation. The theoretical framework of phase aliasing discussed in this paper has shown that there is a reduced influence of noise in unwrapping when IF is low.
Fig. 6. SAR interferometry of Vesuvio area around Naples, Italy (subset of data shown in [11]): (a) amplitude image; (b) interferometrical image (2π corresponds to approximately 175 m in elevation); (c) the terrain mapping obtained from 2-D LMS unwrapping.

Hence, unwrapping can be iterated until the wrapped residual becomes spatially uncorrelated.

APPENDIX A: THRESHOLD IN 2-D ML IF ESTIMATION

An approximate evaluation of threshold effects in nonlinear 2-D ML IF estimation is given. For high SNR the MSE of IF estimation is given by the CR bound for unbiased estimator

\[ \text{var} [\hat{f}(\omega)] = \frac{6}{\gamma N(N-1)} \]  

where a square data matrix totalling \( N \) samples is considered \( (N_x = N_y = \sqrt{N}) \). For low SNR the IF is dominated by threshold effects and the pdf is uniform, giving a variance of \( \pi^2/3 \). For any intermediate value of SNR the MSE \( f_e^2 \) is approximately given by

\[ f_e^2 = p \frac{\pi^2}{3} + (1 - p) \frac{6}{\gamma N(N-1)} \]  

where the probability \( p \) that an outlier occurs is [22]:

\[ p = \frac{1}{N} \sum_{n=2}^{N} \frac{N!((-1)^n \exp \left(-N \gamma \frac{n-1}{n}\right)}}{(N-n)n!} \]  

APPENDIX B: PHASE ALIASING VERSUS NOISE IN PVFD

Let us consider a complex signal \( z_{lm} = a_0 \exp\{j \phi_{lm}\} + w_{lm} \) with uncorrelated zero mean Gaussian noise \( w_{lm} \) \( (a_0 \text{ is constant}) \); the probability density function (pdf) of \( z_{lm} \) is first evaluated and then the pdf of phase aliasing in PVFD as a function of SNR and IF's. Assuming that \( \phi_{lm} = 0 \) at the...
point $(x_l, y_m)$ the phase of $z_{im}$ is

$$\angle z_{im} = \arctan \frac{\text{Im} \{w_{im}\}}{\text{Re} \{w_{im}\}}. \quad (20)$$

Reference [2] has shown that the phase pdf $p_n(\angle z_{im})$ depends on data SNR $\gamma = \sigma_n^2 / 2E[|w_{im}|^2]$ where

$$p_n(\theta) = \exp \left\{ -\gamma \right\} + \frac{\sqrt{\gamma} \cos(\theta)}{2\sqrt{\pi}} \exp \left\{ -\gamma \sin^2(\theta) \right\} \cdot \text{erfc} \left\{ -\sqrt{\gamma} \cos(\theta) \right\}. \quad (21)$$

For SNR below $-10$ dB the pdf becomes uniform while SNR $= 10$ dB represents a lower limit for phase measurements with limited influence of noise. The pdf for $\angle z_{l+m,1}$ is still given by (21) while the pdf of the difference $\angle z_{l+1,m} - \angle z_{l,m}$ is obtained as circular convolution:

$$p_2(\angle z_{l+1,m} - \angle z_{l,m}) = p_n(\angle z_{l+1,m} - \angle z_{l,m}) \ast p_n(\angle z_{l+1,m} - \angle z_{l,m}) \ast \phi_{l,m} \ast \phi_{l,m})$$

where $\phi_{l,m}$ is the effective phase difference or IF for two closely spaced points.

The curl

$$\angle z_{l+1,m} - \angle z_{l,m} = C_{l,m} \equiv \gamma > \pi$$

when $C_{l,m} = \pm 2\pi$ locates the singular points (i.e., the condition of phase aliasing). The relationship (22) indicates that only three phase differences are independent since

$$\left[ \angle z_{l+1,m} - \angle z_{l,m} = C_{l,m} \right]$$

holds true in general.

For high SNR it is most likely that aliasing occurs only on one PVFD in the curl (22) (e.g., $\angle z_{l+1,m} - \angle z_{l,m}$ greater than $\pi$). The probability $p(\gamma, |\phi_{l+1,m} - \phi_{l,m}|)$ that phase difference in each term of the curl is greater than $\pi$ is obtained from (21). Assuming constant IF $\Delta \phi$, the overall probability of 2-D phase aliasing is given by $3p(\gamma, \Delta \phi)$.

For SNR below $10$ dB, it can be shown that the aliasing probability is further complicated by the clustering of singular points. In this case the very low SNR, the three terms in (23) can be assumed as independent, the pdf of the summation of $\angle z_{l+1,m} - \angle z_{l,m}$ is limited in the interval $[-3\pi, +3\pi]$. Since from (23) any value of $|\angle z_{l+1,m} - \angle z_{l,m}| < \pi$ corresponds to the non-aliasing condition, the probability of phase aliasing follows from the evaluation of the probability that the three terms summation is $\geq \pi$ and $\leq -\pi$. The probability of 2-D phase aliasing depends on SNR (singular points caused by noise) as well as on local IF. When phase measurements are dominated by noise (for SNR below $-10$ dB) or IF is close to $\pi$ it can be shown that the aliasing probability is $1/3$.

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